EXTENSIONS OF REPRESENTATIONS OF p-ADIC GROUPS

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Abstract. We calculate extensions between certain irreducible admissible representations of p-adic groups.

To Hiroshi Saito, in memoriam

Contents

1. Introduction

The classification of irreducible admissible representations of groups over local fields has been a very active and successful branch of mathematics. One next step in the subject would be to understand all possible extensions between irreducible representations. Many results of a general kind are known about extensions between admissible representations of p-adic groups, most notably the notion of the Bernstein center and many other results of Bernstein and Casselman. These results reduce the question to one between components of one parabolically induced representation, cf. Lemma 5.1 below. Specific calculations seem not to have attracted attention except for $\mathrm{Ext}^i_G(\mathbb{C},\mathbb{C})$, which is the cohomology $H^i(G,\mathbb{C})$ of G in terms of measurable cochains; besides these, extensions of generalized Steinberg representations are studied in [6, 17]. In this paper, we calculate $\mathrm{Ext}^i_G(\pi_1,\pi_2)$, abbreviated to $\mathrm{Ext}^i(\pi_1,\pi_2)$, between certain irreducible admissible representations π_1,π_2

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of $G = \mathsf{G}(k)$ where G is a connected reductive algebraic group over a non-archimedean local field k of characteristic 0; we abuse notation in the usual way and call G itself a connected reductive algebraic group.

Since extensions of representations of abelian groups are well understood through the cohomology $H^i(\mathbb{Z}^n, \mathbb{C})$ of $\mathbb{Z}^n$, it is no loss of generality when considering extensions $\mathrm{Ext}^i(\pi_1, \pi_2)$ to restrict oneself to the subcategory $\mathcal{R}^\chi(G)$ of the category $\mathcal{R}(G)$ of all smooth representations of G, consisting of those representations on which the center of G acts via a given character χ, which we can also assume to be unitary.

We have two main results. The first is as follows.

Theorem 1. *Let G be a reductive p-adic group over k, and P a maximal k-parabolic subgroup of G with Levi decomposition $P = MN$. Let σ be an irreducible, supercuspidal representation of M, and let $\pi = i_P^G \sigma$, where i_P^G denotes normalized induction. If π is irreducible, then*

$$\mathrm{Ext}^1_{\mathcal{R}^\chi(G)}(\pi, \pi) = \mathbb{C}.$$

If π is reducible, then it has two inequivalent, irreducible subquotients. Let π_1 and π_2 denote these two subquotients. Then

$$\mathrm{Ext}^1_{\mathcal{R}^\chi(G)}(\pi_i, \pi_j) = \begin{cases} 0 & \text{if } i = j, \\ \mathbb{C} & \text{if } i \neq j. \end{cases}$$

Remark 1.1. The theorem is an extension of an observation that one of the authors made concerning reducible unitary principal series representations of $\mathrm{GSp}_4(k)$ arising from the Klingen parabolic (see [18, Remark 11.2]), prompting a similar question for $\mathrm{SL}_2(k)$, which on checking around we found was not known.

A similar statement is true for $(\mathfrak{g}, K)$-modules for representations π_1, π_{-1} of $\mathrm{SL}_2(\mathbb{R})$ of weights $1, -1$ respectively, as follows by looking at the complete list of indecomposable representations of $\mathrm{SL}_2(\mathbb{R})$ supplied by Howe-Tan, cf. [10, Theorem II.1.1.13].

Our second result concerns the components of certain principal series representations of $\mathrm{SL}_n(k)$. Suppose $\omega \colon k^\times \longrightarrow \mathbb{C}^\times$ is a character of order n. We assume that ω is either unramified or totally ramified, in the sense that the restriction of ω to the group $\mathcal{O}^\times$ of units in $k^\times$ is either trivial or has order n. Let π be the principal series representation $\mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$ of $\mathrm{GL}_n(k)$, as well as its restriction to $\mathrm{SL}_n(k)$ which is known to decompose into a direct sum of n inequivalent, irreducible, admissible representations of $\mathrm{SL}_n(k)$, permuted transitively by the action of $\mathrm{GL}_n(k)$ on $\mathrm{SL}_n(k)$ by conjugation. Embed $k^\times$ inside $\mathrm{GL}_n(k)$ as the group of upper left diagonal matrices with all other diagonal entries 1. This $k^\times$ too acts transitively on the set of irreducible summands of the representation π of $\mathrm{SL}_n(k)$; call π_1 one of them. Then the set of irreducible representations of $\mathrm{SL}_n(k)$ appearing in π can be indexed as π_e for e belonging to $k^\times$, in fact more precisely for e belonging to $k^\times/\ker(\omega)$ since it is known that elements of $k^\times$ belonging to $\ker(\omega)$ act trivially on π_1. For the statement of the next theorem, the choice of the base point representation π_1 plays no role, but this indexing of representations occurring in π through $k^\times/\ker(\omega)$ is important.

Let S_π be the group of characters of $k^\times$ generated by ω, i.e., $S_\pi = \{1, \omega, \cdots, \omega^{n-1}\}$. Then the character group $\widehat{S}_\pi$ of S_π can be identified to $k^\times/\ker(\omega)$ via the natural pairing

$$\begin{aligned} S_\pi \times k^\times/\ker(\omega) &\longrightarrow \mathbb{C}^\times, \\ (\chi, x) &\longmapsto \chi(x). \end{aligned}$$

Let $X = \mathbb{C}[S_\pi]$ be the group ring of S_π, and $Y = \mathbb{C}[S_\pi]^0$ be the augmentation ideal of $\mathbb{C}[S_\pi]$. Then Y, and hence $\Lambda^i Y$, are representation spaces of S_π, and it makes sense to talk of $\Lambda^i Y[e]$, the e-th isotypic component of $\Lambda^i Y$, for e a character of S_π, which as mentioned earlier can be identified to $k^\times/\ker(\omega)$.

Theorem 2. *With the notation as above, for $a, b \in k^\times/\ker(\omega)$, we have*

$$\mathrm{Ext}^r(\pi_a, \pi_b) \cong \Lambda^r Y[ba^{-1}].$$

In particular,

$$\mathrm{Ext}^1(\pi_a, \pi_b) = \mathbb{C} \quad \textit{if} \quad a \neq b, \qquad \textit{and} \qquad \mathrm{Ext}^1(\pi_a, \pi_a) = 0.$$

Of course, when $n = 2$, this is a special case of Theorem 1, and thus we have two different computations of extensions of representations of $\mathrm{SL}_2(k)$. Neither is trivial. One uses Kazhdan's orthogonality criterion, and the other uses nontrivial statements about Hecke algebras.

We might add that most of the paper is devoted to the proof of the Theorem 2, which is divided into two separate cases depending on whether the character ω is totally ramified, or is unramified. In both of these cases, the question about calculating the Ext groups is turned into one about modules over appropriate Hecke algebras, then to modules over certain group algebras, and finally to questions about cohomology of groups. Although in these two cases the Hecke algebras involved are quite different, at the end the questions boil down to the same calculation about

$$\mathrm{Ext}^1_{A \rtimes \mathbb{Z}/n}(\chi_1, \chi_2),$$

where A is the group $A = \left\{ (k_1, \cdots, k_n) \in \mathbb{Z}^n \,\middle|\, \sum k_i = 0 \right\}$, with the cyclic permutation action of $\mathbb{Z}/n$ on it, and χ_1, χ_2 are characters of $A \rtimes \mathbb{Z}/n$.

Our Theorem 2 for a very special class of principal series representations of $\mathrm{SL}_n(k)$ begs for a formulation more generally; we offer a conjecture for $\mathrm{SL}_n(k)$:

Conjecture 1.2. *Let π be an irreducible unitary principal series representation of $\mathrm{GL}_n(k)$ induced from a supercuspidal representation σ of a Levi subgroup M. Define,*

$$S_\pi = \{\mu \mid \pi \otimes \mu \cong \pi\},$$

where μ ranges over the set of complex characters of $k^\times$ (considered as characters of $\mathrm{GL}_n(k)$ via the determinant map). Similarly, define,

$$S_\sigma = \{\mu \mid \sigma \otimes \mu \cong \sigma\}.$$

Clearly $S_\sigma \subset S_\pi$, and it is easy to see that S_π/S_σ is a subgroup of the Weyl group $W(\mathrm{GL}_n(k), M) = N_{\mathrm{GL}_n(k)}(M)/M$. Let Y be the character group of $SM = M \cap \mathrm{SL}_n(k)$ which is a module for $W(\mathrm{GL}_n(k), M)$, and in particular for S_π/S_σ. Then characters of S_π —parametrized just as before by a quotient, say Q, of $k^\times$ —determine irreducible representations of $\mathrm{SL}_n(k)$ contained in π, whereas characters of S_π/S_σ determine irreducible representations on $\mathrm{SL}_n(k)$ contained in a principal series representation, say π_0, of $\mathrm{SL}_n(k)$ induced from an irreducible component, say σ_0, of σ restricted to $SM = M \cap \mathrm{SL}_n(k)$. For $a, b \in Q$, we conjecture that:

$$\mathrm{Ext}^r(\pi_a, \pi_b) \cong \Lambda^r Y[ba^{-1}].$$

Remark 1.3. We recall that in the L-packet of $\mathrm{SL}_n(k)$ determined by π, there is a further partitioning depending on whether the representations belong to the same Bernstein component or not: this is the difference between S_π which determines the L-packet and S_π/S_σ which determines the part of the L-packet in a given Bernstein component. The above conjecture includes the statement that unless π_a and π_b belong to the same Bernstein component, all the Ext groups are zero.

Remark 1.4. Although we appeal to existing knowledge about structure of Hecke algebras to prove Theorem 2, some details of the equivalence of the category of representations of p-adic groups versus those of the Hecke algebra are necessary since to convert the problem about representations of p-adic groups to one on Hecke modules, we must know what are the corresponding objects on the Hecke algebra side. It is possible sometimes to come up with the suggested objects on the Hecke algebra with pure thought—for example for Theorem 2 in the totally ramified case, these will be exactly those representations of the Hecke algebra which are of dimension 1, and there are exactly n of them corresponding to components of the principal series representation $\mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$. We have however preferred to identify the modules of the Hecke algebra concretely, and in the process have tried to give an exposition on what goes into it for the benefit of some of the readers, as well as for the authors.

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After the first draft of this paper was written, we saw the recent preprint of Opdam and Solleveld [16]. Our results should emerge as special cases of theirs once appropriate identifications are made, a process that would require some work. However, our proofs are quite different from theirs. The authors thank Opdam and Solleveld for their comments in this regard.

2. Preliminary results for Theorem 1

Given a connected reductive k-group G, and two admissible, finite-length representations π and π' of G having a given central character, one can consider the *Euler-Poincare pairing* between π and π', which is denoted $\mathrm{EP}(\pi, \pi')$, and defined by

$$\mathrm{EP}(\pi, \pi') = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(\pi, \pi').$$

Here, each $\mathrm{Ext}^i(\pi, \pi')$ is a finite-dimensional vector space over $\mathbb{C}$, and is zero when i is greater than the k-split rank of $G/Z(G)$. The notion of the Euler-Poincare pairing and its usefulness in the context of p-adic groups, especially property (d) of Proposition 2.1 below, was noted by Kazhdan in [11]. One can find a proof by Schneider and Stuhler [20] in characteristic zero for property (d); this remains still unresolved in positive characteristic as the convergence of the integral involved is not known in that case.

Proposition 2.1. *Let π and π' be finite-length, smooth representations of a reductive p-adic group G. Then:*

(a) EP *is a symmetric, $\mathbb{Z}$-bilinear form on the Grothendieck group of finite-length representations of G.*
(b) EP *is locally constant. (A family $\{\pi_\lambda\}$ of representations on a fixed vector space V is said to* vary continuously *if all $\pi_\lambda|_K$ are all equivalent for some compact open subgroup K, and the matrix coefficients $\langle \pi_\lambda v, \tilde{v}\rangle$ vary continuously in λ.)*
(c) $\mathrm{EP}(\pi, \pi') = 0$ *if π or π' is induced from any proper parabolic subgroup in G.*
(d) $\mathrm{EP}(\pi, \pi') = \int_{C_{\mathrm{ell}}} \Theta(c)\bar{\Theta}'(c)\, dc$, *where Θ and Θ' are the characters of π and π' assumed to have the same unitary central character, and dc is a natural measure on the set C_{ell} of regular elliptic conjugacy classes in $G/Z(G)$.*

The Euler-Poincare pairing becomes especially useful because of the following two results, concerning vanishing of higher Ext groups and Frobenius reciprocity for Ext.

Proposition 2.2. *Suppose that V in $\mathcal{R}^\chi(G)$ has finite length, and that all of its irreducible subquotients are subquotients of representations induced from supercuspidal representations of a Levi factor of the standard parabolic subgroup P of G, defined by a subset Θ of the set of simple roots. Then*

$\mathrm{Ext}^i_{\mathcal{R}^\chi(G)}(V, V') = 0$ *for* $i > d - |\Theta|$ *and any representation* V' *in* $\mathcal{R}^\chi(G)$ *where* d *is the* k*-split rank of* $G/Z(G)$.

Proof. This is [20, Corollary III.3.3]. □

Proposition 2.3 (Frobenius reciprocity)**.** *Let* P *be a parabolic subgroup of* G *with Levi factorization* $P = MN$*. Let* π *be a smooth representation of* G*, and* σ *a smooth representation of* M*. Then*

$$\mathrm{Ext}^i_{\mathcal{R}^\chi(G)}(\pi, i_P^G(\sigma)) \cong \mathrm{Ext}^i_{\mathcal{R}^\chi(M)}(r_N(\pi), \sigma),$$

where $\mathcal{R}^\chi(M)$ *is the category of smooth representations of* M *on which the center of* G *(which is always contained in* M*) acts via* χ*, and* r_N *denotes the Jacquet functor.*

Proof. This is [4, Theorem A.12]. □

Proposition 2.4. *Let* G *be a reductive* p*-adic group over* k*, and* P *a maximal* k*-parabolic subgroup of* G *with Levi decomposition* $P = MN$*. Let* σ *be an irreducible, supercuspidal representation of* M*, and let* $\pi = i_P^G\sigma$*. If* $N_G(M)/M$ *is nontrivial, it is of order* 2*, in which case write* $N_G(M)/M = \langle w \rangle$*.*

(1) If $N_G(M)/M$ *is trivial,* $\pi = i_P^G\sigma$ *is irreducible.*
(2) If $N_G(M)/M = \langle w \rangle$*, and* $\sigma \ncong \sigma^w$*, then if* π *is reducible, it is indecomposable with distinct Jordan-Hölder factors.*
(3) If $\sigma \cong \sigma^w$*, then by twisting* π *by a character of* G*, we can assume* σ *to be unitary, hence if* π *is reducible, it is completely reducible, and is a direct sum of two distinct irreducible subrepresentations.*

Proof. Part (1) of the proposition is [5, Theorem 7.1.4], and is nontrivial; the other parts are more elementary, and follow from considerations of the Jacquet module which we undertake now. In these parts we do not have to go into the deeper aspects of the subject regarding when reducibility actually occurs.

For $P = MN$, let $P^- = MN^-$ be the opposite parabolic. Then P^- and P are conjugate in G if and only if $N_G(M) \neq M$. If P and P^- are conjugate in G, then P is the unique parabolic in G up to conjugacy in its associate class; otherwise, there are two distinct conjugacy classes of parabolics in the associate class of P. It follows from the *geometric lemma* that $r_N(\pi) = \sigma$ if $N_G(M) = M$, and that if $N_G(M) \neq M$, then $r_N(\pi)$ has Jordan-Hölder factors σ and σ^w. If $\sigma \ncong \sigma^w$, then since σ is supercuspidal, $r_N(\pi) = \sigma \oplus \sigma^w$. In this case if π is reducible, with π_1 and π_2 as the Jordan-Hölder factors of π, then we can assume that $r_N(\pi_1) = \sigma$, and $r_N(\pi_2) = \sigma^w$. From Frobenius reciprocity,

$$\mathrm{Hom}_G[\pi_2, \pi] = \mathrm{Hom}_M[r_N(\pi_2), \sigma] = \mathrm{Hom}_M[\sigma^w, \sigma] = 0.$$

proving that if $\sigma \ncong \sigma^w$, and π is reducible, it is indecomposable with distinct Jordan-Hölder factors, proving part (2) of the proposition.

Note that if $N_G(M)/M$ is nontrivial and $\sigma^w \cong \sigma$, σ must be unitary when restricted to the intersection of M and the derived group $[G, G]$ of G. If the supercuspidal representation σ of the Levi subgroup M is unitary, then π is completely reducible, and we see that the Jordan-Hölder factors of π are distinct by a calculation of $\mathrm{Hom}_G[\pi, \pi] = \mathrm{Hom}_M[r_N(\pi), \sigma]$, which is a two dimensional vector space over $\mathbb{C}$. If σ is not unitary when restricted to $M \cap [G, G]$, in particular $\sigma \ncong \sigma^w$, we see that the Jordan-Hölder factors of π are distinct by noting that their Jacquet modules are σ and σ^w, proving part (3) of the proposition. □

3. Proof of Theorem 1

Proof. If π is irreducible, then the result follows from Proposition 2.1(c), together with Proposition 2.2.

Suppose from now on that π is reducible. Assume first that we have a non-split short exact sequence

$$(*) \qquad 0 \longrightarrow \pi_1 \longrightarrow \pi \longrightarrow \pi_2 \longrightarrow 0.$$

From $(*)$, we have that $\mathrm{Ext}^1(\pi_2, \pi_1)$ is nontrivial, and this by Proposition 2.4 implies that the inducing representation σ is not unitary even after twisting by characters of G (restricted to the Levi subgroup). By replacing the inducing representation σ with its Weyl conjugate, we obtain another principal series representation π' which will have π_1 as a quotient, and π_2 as a subrepresentation. Since σ is not unitary, Proposition 2.4 implies that π' does not split. Thus, $\mathrm{Ext}^1(\pi_1, \pi_2)$ is nontrivial.

Working in the category $\mathcal{R}^\chi(G)$, apply $\mathrm{Hom}(\pi_1, -)$ to $(*)$ and consider the induced long exact sequence

$$\begin{array}{llll} 0 & \longrightarrow \mathrm{Hom}(\pi_1, \pi_1) & \longrightarrow \mathrm{Hom}(\pi_1, \pi) & \longrightarrow \mathrm{Hom}(\pi_1, \pi_2) \\ & \longrightarrow \mathrm{Ext}^1(\pi_1, \pi_1) & \longrightarrow \mathrm{Ext}^1(\pi_1, \pi) & \longrightarrow \mathrm{Ext}^1(\pi_1, \pi_2) \\ & \longrightarrow \mathrm{Ext}^2(\pi_1, \pi_1) & \longrightarrow \cdots & \end{array}$$

By Proposition 2.2, $\mathrm{Ext}^2(\pi_1, \pi_1) = 0$. From Proposition 2.4, $\mathrm{Hom}(\pi_1, \pi_2) = 0$, so we have a short exact sequence

$$(\triangle) \qquad 0 \longrightarrow \mathrm{Ext}^1(\pi_1, \pi_1) \longrightarrow \mathrm{Ext}^1(\pi_1, \pi) \longrightarrow \mathrm{Ext}^1(\pi_1, \pi_2) \longrightarrow 0.$$

Since $\mathrm{Ext}^1(\pi_1, \pi_2)$ is nonzero, and since $r_N\pi_1 \cong \sigma$, Frobenius reciprocity (Proposition 2.3) gives

$$\mathrm{Ext}^1_{\mathcal{R}^\chi(G)}(\pi_1, \pi) \cong \mathrm{Ext}^1_{\mathcal{R}^\chi(M)}(r_N\pi_1, \sigma) \cong \mathrm{Ext}^1_{\mathcal{R}^\chi(M)}(\sigma, \sigma).$$

Let χ_σ denote the central character of σ. Then σ is projective in $\mathcal{R}^{\chi_\sigma}(M)$. Since $Z(M)/Z(G)$ has split rank 1, $\dim \mathrm{Ext}^1_{\mathcal{R}^\chi(M)}(\sigma, \sigma) = 1$. (This amounts to the assertion that $\mathrm{Ext}^1_{k^\times}(\mu, \mu) = \mathbb{C}$, where μ is a one-dimensional character of $k^\times$.) From $(\triangle)$, we thus have that $\mathrm{Ext}^1(\pi_1, \pi_1) = 0$ and $\mathrm{Ext}^1(\pi_1, \pi_2) = \mathbb{C}$, as desired.

We now turn to the case when $\pi = \pi_1 + \pi_2$. This is the nontrivial part of the proposition, where one wants to construct a nontrivial extension between π_1 and π_2, even though the extension afforded by the principal series representation in which they sit is split.

From Proposition 2.2, $\mathrm{Ext}^i(\pi_1, \pi_1) = 0$ for $i > 1$. From [1, Proposition 2.1(c)], the character of π_1 does not vanish on the elliptic set. By Proposition 2.1(d), $\mathrm{EP}(\pi_1, \pi_1)$ is positive. Thus,

$$\begin{aligned} \dim \mathrm{Ext}^1(\pi_1, \pi_1) &= \dim \mathrm{Hom}(\pi_1, \pi_1) - \mathrm{EP}(\pi_1, \pi_1) \\ &= 1 - \mathrm{EP}(\pi_1, \pi_1) < 1, \end{aligned}$$

and thus $\mathrm{Ext}^1(\pi_1, \pi_1) = 0$. From Proposition 2.1(c), $\dim \mathrm{Ext}^1(\pi_1, \pi) = 1$, so it follows that $\dim \mathrm{Ext}^1(\pi_1, \pi_2) = 1$. The rest of the proposition follows by symmetry between π_1 and π_2. □

Remark 3.1. It may be worth emphasizing that although the proof of Theorem 1 above might look straightforward, it uses rather deep Proposition 2.1(d) which is known only in characteristic 0, and hence so also this theorem.

4. A construction of Savin

If π is a reducible unitary principal series representation of $\mathrm{SL}_2(k)$ then it has two inequivalent, irreducible subquotients π_1 and π_2. By Theorem 1 we know that

$$\mathrm{Ext}^1_{\mathrm{SL}_2(k)}(\pi_1, \pi_2) = \mathbb{C}.$$

G. Savin has offered a natural construction of such an extension, at least when π arises from an unramified quadratic character of $k^\times$. This construction may be useful in many similar situations, so we outline it, referring to [19] for details. We begin with some generality.

Let K be an open compact subgroup of a split reductive p-adic group G. Let $\mathcal{H} = C_c(K\backslash G/K)$ be the Hecke algebra of K-bi-invariant compactly supported functions on G. If V is a smooth G-module, then V^K is a left $\mathcal{H}$-module. It is a standard fact that if V is an irreducible G-module, and if V^K is non-zero, the latter is an irreducible $\mathcal{H}$-module. Conversely, if E is a left $\mathcal{H}$-module then

$$I(E) := C_c(G/K) \otimes_{\mathcal{H}} E$$

is a smooth G-module. As a right $\mathcal{H}$-module, $C_c(G/K)$ can be decomposed as

$$C_c(G/K) = C_c(G/K)' \oplus \mathcal{H},$$

where $C_c(G/K)'$ denotes the sum of all non-trivial left K-submodules of $C_c(G/K)$. It follows that $I(E)^K \cong E$, as $\mathcal{H}$-modules. Note that $I(E)^K$ generates the G-module $I(E)$. Let $U(E) \subseteq I(E)$ be the sum of all G-submodules of $I(E)$ intersecting $I(E)^K$ trivially. Let $J(E)$ be the quotient $I(E)/U(E)$. Then $J(E)$ is generated by $J(E)^K \cong E$, and any submodule of $J(E)$ contains non-zero K-fixed vectors. Using this, the following proposition is proved.

Proposition 4.1. *Let E be an irreducible $\mathcal{H}$-module. Then $J(E)$ is the unique irreducible quotient of $I(E)$.*

Assume now that K is hyperspecial and let $\mathcal{I} \subseteq K$ be an Iwahori subgroup. Since $\mathcal{H}$ is commutative, every irreducible $\mathcal{H}$-module is one dimensional. Pick one, and call it $\mathbb{C}_\chi$. Any subquotient of $I(\mathbb{C}_\chi)$ is generated by its $\mathcal{I}$-fixed vectors. As in [19], denoting by X the cocharacter group of a maximal split torus of G, we have

$$I(\mathbb{C}_\chi)^{\mathcal{I}} = C_c(\mathcal{I}\backslash G/K) \otimes_{\mathcal{H}} \mathbb{C}_\chi \cong \mathbb{C}[X] \otimes_{\mathbb{C}[X]^W} \mathbb{C}_\chi.$$

From generality about integral extensions of commutative integrally closed domains, $\mathbb{C}[X] \otimes_{\mathbb{C}[X]^W} \mathbb{C}_\chi$ has dimension equal to $|W|$, hence $\dim(I(\mathbb{C}_\chi)^{\mathcal{I}}) = |W|$. We specialize further to $G = \mathrm{SL}_2(k)$. Let $V = \pi_1$ be the unique irreducible tempered representation of G such that $\dim(V^K) = \dim(V^{\mathcal{I}}) = 1$. Then $I(V^K)$ has length 2, and is the representation of $\mathrm{SL}_2(k)$ corresponding to a non-trivial element of $\mathrm{Ext}^1_{\mathrm{SL}_2(k)}(\pi_1, \pi_2) = \mathbb{C}$ that we desired to construct since the unique irreducible quotient of $I(V^K)$ is $V = \pi_1$. If U is the unique irreducible submodule of $I(V^K)$, then $\dim(U^K) = 0$ and $\dim(U^{\mathcal{I}}) = 1$, thus U is the irreducible representation such that $V \oplus U$ is isomorphic to the representation induced from the unique non-trivial, unramified, quadratic character of $k^\times$, forcing U to be π_2.

5. Preliminary results for Theorem 2

We recall a small part of the theory of types [3]. The starting point is the fundamental result, due to Bernstein, that the category $\mathcal{R}(G)$ of smooth complex representations of G decomposes as a direct sum of certain indecomposable full subcategories, now often called the Bernstein components of $\mathcal{R}(G)$:

$$\mathcal{R}(G) = \coprod_{\mathfrak{s} \in \mathcal{B}(G)} \mathcal{R}_{\mathfrak{s}}(G).$$

The indexing set $\mathcal{B}(G)$ consists of (equivalence classes of) irreducible supercuspidal representations of Levi subgroups M of G up to conjugation by G and twisting by unramified characters of M, i.e., characters that are trivial on all compact subgroups of M.

Suppose $\mathfrak{s} \in \mathcal{B}(G)$ corresponds to an irreducible supercuspidal representation, say σ, of a Levi subgroup M of G. The irreducible objects in $\mathcal{R}_{\mathfrak{s}}(G)$ are then precisely the irreducible subquotients of the various parabolically induced representations $i_P^G(\sigma\nu)$ as ν varies through the unramified characters of M and where P is any parabolic subgroup of G with Levi component M.

We note the following lemma which is a simple consequence of Bernstein theory but which, however, does not follow from Frobenius reciprocity. The result can also be found in [23, Theorem 6.1].

Lemma 5.1. *Let π_1 and π_2 be two irreducible admissible representations of G with different cuspidal support. Then*

$$\mathrm{Ext}^i(\pi_1, \pi_2) = 0 \quad \text{for all} \quad i \geq 0.$$

Proof. If π_1 and π_2 belong to different Bernstein components, then there is nothing to prove. If they belong to the same Bernstein component, then associated to the component is an irreducible affine algebraic variety over $\mathbb{C}$ whose space of regular functions is the center of the corresponding category. Now given two distinct points on the affine algebraic variety corresponding to π_1 and π_2, there is an element, call it f, in the center of the category such that f acts by 0 on π_1, and by 1 on π_2. Standard homological algebra then proves that $\mathrm{Ext}^i(\pi_1, \pi_2) = 0$ for all $i \geq 0$. □

A pair (K, ρ) consisting of a compact open subgroup K of G and a smooth irreducible representation ρ of K is called an $\mathfrak{s}$-*type* if the irreducible smooth representations of G that contain ρ on restriction to K are exactly the irreducible objects in $\mathcal{R}_\mathfrak{s}(G)$. In this case, the category $\mathcal{R}_\mathfrak{s}(G)$ is equivalent to the category of (left) modules over the intertwining or Hecke algebra of ρ. More precisely, let W denote the space of ρ and write $\mathcal{H}(G, \rho)$ for the space of compactly supported functions $\Phi : G \to \mathrm{End}(W^\vee)$ such that

$$\Phi(k_1 g k_2) = \rho^\vee(k_1)\Phi(g)\rho^\vee(k_2),$$

where, as usual, $\rho^\vee$ is the dual of ρ. This is a convolution algebra (with respect to a fixed Haar measure on G). The endomorphism algebra $\mathrm{End}_G(\mathrm{ind}_K^G \rho)$ is isomorphic to the opposite of the algebra $\mathcal{H}(G, \rho)$, so that a right $\mathrm{End}_G(\mathrm{ind}_K^G \rho)$-module is naturally a left $\mathcal{H}(G, \rho)$-module. This allows one to give a natural $\mathcal{H}(G, \rho)$-module structure on $\mathrm{Hom}_K(\rho, \pi)$ for any smooth representation π of G.

We mention two basic examples of $\mathfrak{s}$-types which served as precursors to the general theory. In the first, $M = G$. Thus σ is a supercuspidal representation of G and the irreducible objects in $\mathcal{R}_\mathfrak{s}(G)$ are simply the unramified twists of σ. In this case, elementary arguments show that the existence of an $\mathfrak{s}$-type is closely related to the statement that σ is induced from a compact mod center subgroup of G (see [3, §5.4]). In particular, if σ is induced in this way, then an $\mathfrak{s}$-type exists and is easily described in terms of the inducing data for σ. We note that through the work of J.-K. Yu [24], Julee Kim [12], and S. Stevens [22], the existence of such types is now known for all reductive groups under a tameness hypothesis and for many classical groups in odd residual characteristic. Types exist for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ without any restriction on residue characteristic by the work of Bushnell-Kutzko [3], and Goldberg-Roche [7,8].

In the second example, $\mathcal{R}_\mathfrak{s}(G)$ is defined by σ, the trivial representation of a minimal Levi subgroup M of G. Since a minimal Levi subgroup has no proper parabolic subgroup, the trivial representation of M is supercuspidal; further, it is known that M is compact modulo its center. In this case, the trivial representation of an Iwahori subgroup $\mathcal{I}$ provides an $\mathfrak{s}$-type: this is the classical result of Borel and Casselman that an irreducible smooth representation of G contains non-trivial $\mathcal{I}$-fixed vectors if and only if it is a constituent of an unramified principal series. The general theory posits that these two examples are extreme instances of a general phenomenon.

A fundamental feature of Bushnell-Kutzko's theory is that parabolic induction can be transferred effectively to the Hecke algebra setting and we make essential use of this feature below. We recall a special case which is more than adequate to our needs. Let σ be an irreducible supercuspidal representation of a Levi subgroup M of G and write $\mathcal{R}_{\mathfrak{s}_M}(M)$ for the resulting component of $\mathcal{R}(M)$. Thus the irreducible objects in $\mathcal{R}_{\mathfrak{s}_M}(M)$ are simply the various unramified twists of σ. We also write $\mathcal{R}_{\mathfrak{s}}(G)$ for the resulting component of $\mathcal{R}(G)$. We assume that $\mathcal{R}_{\mathfrak{s}_M}(M)$ admits a type (K_M, ρ_M). We assume also that (K_M, ρ_M) admits a *G-cover* (K, ρ) whose definition due to Bushnell and Kutzko we recall below (see [3] §8).

Given a parabolic $P = MN$, with opposite parabolic $P^- = MN^-$, we call a pair (J, τ) consisting of a compact open subgroup J of G, and a finite-dimensional irreducible representation τ of J *decomposed* with respect to (P, M) if

(1) $J = (J \cap N^-) \cdot (J \cap M) \cdot (J \cap N)$.
(2) The groups $J \cap N^-$ and $J \cap N$ act trivially under τ, so τ restricted to $J_M = J \cap M$ is an irreducible representation; call it τ_M.

Let $I_G(\tau)$ denote the set of elements g in G such that there is a function f in $\mathcal{H}(G, \tau)$ whose support contains g. It can be seen that if (J, τ) is decomposed with respect to (P, M), then

$$I_M(\tau_M) = I_G(\tau) \cap M.$$

Further, if $\phi \in \mathcal{H}(M, \tau_M)$ has support $J_M z J_M$ for some $z \in M$, there is a unique $T(\phi) = \Phi \in \mathcal{H}(G, \tau)$ with support contained in JzJ, and with $\Phi(z) = \phi(z)$. The map $T \colon \phi \to \Phi$ from $\mathcal{H}(M, \tau_M)$ to $\mathcal{H}(G, \tau)$ is an isomorphism of vector spaces onto $\mathcal{H}(G, \tau)_M$, the subspace of $\mathcal{H}(G, \tau)$ with support contained in JMJ.

One calls an element $z \in M$ *positive* with respect to (J, N) if it satisfies,

$$z(J \cap N)z^{-1} \subset (J \cap N), \;\; z^{-1}(J \cap N^-)z \subset (J \cap N^-).$$

Let I^+ denote the set of positive elements of $I_M(\tau_M) = I_G(\tau) \cap M$, and let $\mathcal{H}(M, \tau_M)^+$ denote the space of functions in $\mathcal{H}(M, \tau_M)$ with support contained in $J_M I^+ J_M$. Then the map T from $\mathcal{H}(M, \tau_M)$ to $\mathcal{H}(G, \tau)$ when restricted to $\mathcal{H}(M, \tau_M)^+$ is an algebra homomorphism sending identity element of $\mathcal{H}(M, \tau_M)$ to the identity element of $\mathcal{H}(G, \tau)$; it extends uniquely to an injective algebra homomorphism from $\mathcal{H}(M, \tau_M)$ to $\mathcal{H}(G, \tau)$ when the pair (J, τ) is a G-cover (to be defined below) of (J_M, τ_M).

Define an element ζ of the center $Z(M)$ of M to be *strongly positive* if it is positive, and has the property that given compact open subgroups H_1 and H_2 of N, there is a power ζ^m, $m \geq 0$, which conjugates H_1 into H_2, and similarly a property for subgroups of N^- by negative powers of ζ.

Here then is the definition of a G-cover.

Definition 5.2. Let M be a Levi subgroup of a reductive group G. Let J_M be a compact open subgroup of M, and (τ_M, W) an irreducible smooth representation of J_M. Let J be a compact open subgroup of G, and τ an irreducible smooth representation of J. The pair (J, τ) is G-cover of (J_M, τ_M) if the following holds:

(1) The pair (J, τ) is decomposed with respect to (M, P), in the sense defined earlier, for all parabolics P with Levi M.
(2) $J \cap M = J_M$, and $\tau|_{J_M} = \tau_M$.
(3) For every parabolic $P = MN$ with Levi M, there exists an invertible element of $\mathcal{H}(G, \tau)$ supported on a double coset $J\zeta_P J$ where $\zeta_P \in Z(M)$ is strongly (J, N)-positive.

The definition of a G-cover is tailored to achieve the following result, which can be found in [3].

Proposition 5.3. *Let P be a parabolic subgroup of a reductive k-group G, and let M be a Levi factor of P. Let J_M be a compact open subgroup of M, and (τ_M, W) an irreducible smooth representation of J_M. Suppose that $\mathcal{R}_{\mathfrak{s}_M}(M)$ is a component of $\mathcal{R}(M)$, defined by the type (τ_M, W). Let J be a compact open subgroup of G, and τ an irreducible smooth representation of J. If the pair (J, τ) is a G-cover of (J_M, τ_M), then parabolic induction from P to G of representations in $\mathcal{R}_{\mathfrak{s}_M}(M)$ defines a component in $\mathcal{R}(G)$ with (J, τ) as a type.*

Recall the following result of Moy-Prasad, [15, Proposition 6.4]. Let $\mathbb{P}$ be a parahoric subgroup of a reductive group G over k, with $\mathbb{P}^+$ the pro-unipotent radical of $\mathbb{P}$. If $\mathbb{F}_q$ is the residue field of k, then $\mathbb{P}/\mathbb{P}^+$ is the group of rational points of a reductive $\mathbb{F}_q$-group. There is a unique $\mathbb{P}$-conjugacy class of Levi subgroups M in G such that $\mathbb{M} = \mathbb{P} \cap M$ is a maximal parahoric subgroup in M with

$$\mathbb{M}/\mathbb{M}^+ \cong \mathbb{P}/\mathbb{P}^+.$$

The following result of Morris [14] constructs G-covers for all depth-zero types of Levi subgroups. The relevance of this result for us is that in the tame case, i.e., $(n, p) = 1$, the representations of $\mathrm{SL}_n(k)$ that we consider have depth zero. Although we will obtain G-covers for them from the work of Goldberg-Roche, in the tame case we could have used Morris's result instead. In fact, Morris goes further to identify the Hecke algebra $\mathcal{H}(G, \rho)$ too, but we do not go into that.

Proposition 5.4. *Let G be a reductive algebraic group over a non-archimedean local field k. Let $\mathbb{P}$ be a parahoric subgroup of G, defining a Levi subgroup M, and maximal parahoric $\mathbb{M}$ in M as above with*

$$\mathbb{M}/\mathbb{M}^+ \cong \mathbb{P}/\mathbb{P}^+,$$

allowing one to construct representations of $\mathbb{P}$ from representations of $\mathbb{M}/\mathbb{M}^+$. Let ρ be any irreducible representation of $\mathbb{P}$ arising out of this construction. Then $(\mathbb{P}, \rho)$ is a G-cover of $(\mathbb{M}, \rho|_{\mathbb{M}})$.

Let P be a parabolic subgroup of G with Levi component M. The functor i_P^G of normalized parabolic induction from $\mathcal{R}(M)$ to $\mathcal{R}(G)$ takes $\mathcal{R}_{\mathfrak{s}_M}(M)$ to $\mathcal{R}_{\mathfrak{s}}(G)$. It therefore corresponds, under the equivalence of $\mathcal{R}_{\mathfrak{s}}(G)$ with $\mathcal{H}(G,\rho)$-modules and its analogue for M, to a certain functor from $\mathcal{H}(M,\rho_M)$-Mod to $\mathcal{H}(G,\rho)$-Mod. To describe this, we note that there is a certain (explicit) embedding of $\mathbb{C}$-algebras

$$\lambda_P\colon \mathcal{H}(M,\rho_M) \longrightarrow \mathcal{H}(G,\rho).$$

This induces a functor $(\lambda_P)*$ from $\mathcal{H}(M,\rho_M)$-Mod to $\mathcal{H}(G,\rho)$-Mod, given on objects by

$$S \longmapsto \mathrm{Hom}_{\mathcal{H}(M,\rho_M)}(\mathcal{H}(G,\rho), S)$$

where $\mathcal{H}(G,\rho)$ is viewed as a left $\mathcal{H}(M,\rho_M)$-module via λ_P and $\mathcal{H}(G,\rho)$ acts by right translations. We have the following commutative diagram (up to natural equivalence) by [3, Cor. 8.4]:

$$\begin{array}{ccc} \mathcal{R}_{\mathfrak{s}}(G) & \xrightarrow{\simeq} & \mathcal{H}(G,\rho)\text{-Mod} \\ {\scriptstyle i_P^G}\big\uparrow & & \big\uparrow{\scriptstyle (\lambda_P)*} \\ \mathcal{R}_{\mathfrak{s}_M}(M) & \xrightarrow{\simeq} & \mathcal{H}(M,\rho_M)\text{-Mod}. \end{array} \tag{5.1}$$

In other words, normalized parabolic induction from $\mathcal{R}_{\mathfrak{s}_M}(M)$ to $\mathcal{R}_{\mathfrak{s}}(G)$ corresponds to $(\lambda_P)*$ under the equivalences of the theory of types. (Note although [3] explicitly treats only unnormalized induction, it is a trivial matter to adjust the arguments so that they apply to normalized induction.)

6. Proof of Theorem 2 in the totally ramified case

We set $G = SL_n(k)$. Let T denote the standard split torus of diagonal elements in G and T^1 the unique maximal compact subgroup of T. We write

$$A = \Big\{ (a_1, \cdots, a_n) \in \mathbb{Z}^n \;\Big|\; \sum a_i = 0 \Big\}.$$

Fix a uniformizer ϖ in k. Consider the map $a \mapsto \varpi^a\colon A \to T$ where

$$\varpi^a = \mathrm{diag}(\varpi^{a_1}, \ldots, \varpi^{a_n}), \quad \text{for} \quad a = (a_1, \ldots, a_n).$$

This splits the inclusion $T^1 \hookrightarrow T$, i.e.,

$$(t_1, a) \mapsto t_1\varpi^a\colon T^1 \times A \xrightarrow{\simeq} T \tag{6.1}$$

is an isomorphism. In this way, we can view characters of T as pairs consisting of characters of T^1 and characters of A (equivalently, unramified characters of T).

Let $\omega\colon k^\times \to \mathbb{C}^\times$ be a character of order n such that the restriction of ω to $\mathcal{O}^\times$, call it $\omega_{\mathcal{O}}$, remains of order n, where $\mathcal{O}$ denotes the ring of integers in k. Let χ be the character of T^1 given by

$$\chi(\mathrm{diag}(x_1, x_2, \ldots, x_n)) = \omega_{\mathcal{O}}(x_1)\omega_{\mathcal{O}}(x_2)^2 \cdots \omega_{\mathcal{O}}(x_n)^n.$$

We are interested in the resulting Bernstein component $\mathcal{R}_\chi(G)$. The irreducible objects in this component consist of the irreducible subquotients of the family of induced representations $i_B^G(\chi\nu)$ as ν varies through the unramified characters of T (and B is any Borel subgroup containing T).

We write $\mathcal{R}_\chi(T)$ for the Bernstein component of T determined by χ. The irreducible objects in $\mathcal{R}_\chi(T)$ are simply the various extensions of χ to T. It is obvious that (T^1, χ) is a type for $\mathcal{R}_\chi(T)$. By [7], there is a G-cover (K, ρ) of (T^1, χ) which is therefore a type for $\mathcal{R}_\chi(G)$. If ω is trivial on $1 + \varpi\mathcal{O}$, then K is the Iwahori subgroup of $\mathrm{SL}_n(k)$. If ω is trivial on $1 + \varpi^n\mathcal{O}$ but not on $1 + \varpi^{n-1}\mathcal{O}$, for $n > 1$, then

$$K = N^-([(n+1)/2]) \cdot T(\mathcal{O}) \cdot N([n/2]),$$

where $[x]$ denotes the integral part of the rational number x; and $N(i)$ (resp. $N^-(i)$) denotes the group of upper- (resp. lower-) triangular unipotent matrices with non-diagonal entries in $\varpi^i\mathcal{O}$. The restriction of ρ to $T(\mathcal{O})$ is the character $(1, \omega, \cdots, \omega^{n-1})$.

We next describe the Hecke algebra $\mathcal{H}(G, \rho)$ and the algebra embedding $\lambda_B : \mathcal{H}(T, \chi) \to \mathcal{H}(G, \rho)$ (for B a fixed Borel containing T).

To simplify some formulas, we take convolution in $\mathcal{H}(T, \chi)$ (resp. $\mathcal{H}(G, \rho)$) with respect to the Haar measure that gives T^1 (resp. K) unit measure. For $a \in A$, let ϕ_a denote the the unique function in $\mathcal{H}(T, \chi)$ with support $T^1\varpi^a$ such that $\phi_a(\varpi^a) = 1$. The assignment $\phi_a \mapsto a$, for $a \in A$, clearly extends to a $\mathbb{C}$-algebra isomorphism $\mathcal{H}(T, \chi) \simeq \mathbb{C}[A]$.

Proposition 6.1. *Let $H = A \rtimes \mathbb{Z}/n$ where $\mathbb{Z}/n$ acts on A by cyclic permutation of the co-ordinates. Then there is a $\mathbb{C}$-algebra isomorphism $\mathcal{H}(G, \rho) \simeq \mathbb{C}[H]$, the complex group algebra of H. This fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{H}(G,\rho) & \xrightarrow{\simeq} & \mathbb{C}[H] \\ {\scriptstyle \lambda_B}\uparrow & & \uparrow \\ \mathcal{H}(T,\chi) & \xrightarrow{\simeq} & \mathbb{C}[A] \end{array} \tag{6.2}$$

in which the right vertical arrow is the obvious inclusion and the bottom horizontal arrow is the isomorphism that sends ϕ_a to a, for $a \in A$.

Proof. Theorem 11.1 of [8] gives the following description of $\mathcal{H}(G, \rho)$. First, for $a \in A$, we set $\Phi_a = \lambda_B(\phi_a)$ so that

$$\Phi_a\Phi_{a'} = \Phi_{a+a'},$$

for all $a, a' \in A$. Writing w for the cycle $(1\,2 \ldots n)$, there is a special function $\Phi_w \in \mathcal{H}(G, \rho)$ that satisfies

(1) $\Phi_w^n = \Phi_0$, the identity element of $\mathcal{H}(G, \rho)$,
(2) $\Phi_w\Phi_a\Phi_w^{-1} \doteq \Phi_{w(a)}$, for all $a \in A$.

Here w acts on A in the obvious way (by cyclic permutation of the co-ordinates), and $\doteq$ denotes equality up to multiplication by scalars. (Note that it follows from (2) that the order of Φ_w is exactly n.) Finally, $\mathcal{H}(G, \rho)$ is generated as a $\mathbb{C}$-algebra by Φ_w and the elements Φ_a, for $a \in A$.

To prove the Proposition, we will show that (2) is actually an equality. For this, we consider the induced representation $i_B^G(\chi)$ (viewing χ as a character of T that is trivial on A). This decomposes as a sum of n distinct irreducible subrepresentations. This can be seen by noting the following:

- A unitary principal series representation of $\mathrm{GL}_n(k)$ is irreducible.
- An irreducible admissible representation π of $\mathrm{GL}_n(k)$ when restricted to $\mathrm{SL}_n(k)$ decomposes as a sum of a finite collection of irreducible representations whose cardinality is the same as the cardinality of self-twists of π:

$$\{\alpha\colon k^\times \to \mathbb{C}^\times | \pi \otimes \alpha \cong \pi\} = \{1, \omega, \cdots, \omega^{n-1}\}.$$

We now appeal to the diagram (5.1). The $\mathcal{H}(G,\rho)$-module that corresponds to $i_B^G(\chi)$ has dimension n and so must split as a sum of n one-dimensional submodules. Note that each ϕ_a, for $a \in A$, acts trivially on the $\mathcal{H}(T,\chi)$-module corresponding to the character χ of T. It follows easily that there is a $\mathbb{C}$-algebra homomorphism $\Lambda : \mathcal{H}(G,\rho) \to \mathbb{C}$ such that $\Lambda(\Phi_a) = 1$, for all $a \in A$ (in fact, there are n such homomorphisms). Applying Λ to (2), we see that (2) must be an equality. □

Combining (6.2) (or more properly the diagram induced by (6.2) on module categories) and (5.1), we obtain a commutative diagram of functors (up to equivalence)

$$\begin{array}{ccc} \mathcal{R}_\chi(G) & \xrightarrow{\simeq} & \mathbb{C}[H]\text{-Mod} \\ {\scriptstyle i_B^G}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{R}_\chi(T) & \xrightarrow{\simeq} & \mathbb{C}[A]\text{-Mod}. \end{array} \tag{6.3}$$

Explicitly, if M is a $\mathbb{C}[A]$-module, then $i(M) = \mathrm{Hom}_{\mathbb{C}[A]}(\mathbb{C}[H], M)$ where, as above, the $\mathbb{C}[H]$-action is given by right translations. Let ν be an unramified character of T viewed as a character of A via $a \mapsto \nu(\varpi^a)$. The bottom horizontal arrow takes the object $\chi\nu$ in $\mathcal{R}_\chi(T)$ to the simple $\mathbb{C}[A]$-module $\mathbb{C}_\nu$ corresponding to ν.

We are interested in a particular family of induced representations in $\mathcal{R}_\chi(G)$. To describe this family, let ω be an n-th root of unity and write ν_ω for the unramified character of T given by

$$\nu_\omega(\varpi^a) = \omega^{a_1}\omega^{2a_2}\cdots\omega^{na_n},$$

for $a = \mathrm{diag}(a_1, \ldots, a_n)$. To simplify the notation, we write $\mathbb{C}_\omega$ in place of $\mathbb{C}_{\nu_\omega}$ for the $\mathbb{C}[A]$-module corresponding to ν_ω. By (6.3), the induced representation $i_B^G(\chi\nu_\omega)$ corresponds to the $\mathbb{C}[H]$-module $i(\mathbb{C}_\omega)$.

Observe that $\mathbb{C}_\omega$ is fixed under the action of $\mathbb{Z}/n$ on A (by cyclic permutations). Indeed,

$$\begin{aligned}(k_n, k_1, \ldots, k_{n-1}) &\mapsto \omega^{k_n}\omega^{2k_1}\cdots\omega^{nk_{n-1}} \\ &= \omega^{k_1}\omega^{k_2}\cdots\omega^{k_n}\left(\omega^{k_1}\omega^{2k_2}\cdots\omega^{nk_n}\right) \\ &= \omega^{k_1}\omega^{2k_2}\cdots\omega^{nk_n}.\end{aligned}$$

It follows that for any character $\eta : \mathbb{Z}/n \to \mathbb{C}^\times$ the $\mathbb{C}[A]$-module $\mathbb{C}_\omega$ extends to a $\mathbb{C}[H]$ module $\mathbb{C}_{\omega,\eta}$ in which $\mathbb{Z}/n$ acts by η and

$$i(\mathbb{C}_\omega) = \bigoplus_\eta \mathbb{C}_{\omega,\eta},$$

as η varies through the distinct characters of $\mathbb{Z}/n$.

To finish, we therefore need to determine $\mathrm{Ext}^i_{\mathbb{C}[H]}(\mathbb{C}_{\omega,\eta}, \mathbb{C}_{\omega,\eta'})$ for all characters η, η' of $\mathbb{Z}/n$. We have

$$\begin{aligned}\mathrm{Ext}^i_{\mathbb{C}[H]}(\mathbb{C}_{\omega,\eta}, \mathbb{C}_{\omega,\eta'}) &\simeq \mathrm{Ext}^i_{\mathbb{C}[H]}(\mathbb{C}_{1,1}, \mathbb{C}_{1,\eta'\eta^{-1}}) \\ &= H^i(H, \mathbb{C}_{1,\eta'\eta^{-1}}).\end{aligned}$$

Of course, $\mathbb{C}_{1,\eta'\eta^{-1}}$ is a character of H that is trivial on A. To compute these cohomology groups, we use the following general result.

Lemma 6.2. *Let N be a finite-index normal subgroup of a group G, and V a $\mathbb{C}[G]$-module. Then*

$$H^i(G, V) \cong H^i(N, V)^{G/N}.$$

Proof. This follows from the spectral sequence which calculates cohomology of G in terms of that of N after we have noted that since G/N is finite, it has no cohomology in positive degree for a coefficient system which is a $\mathbb{C}$-vector space. □

Corollary 6.3. *Let N be a normal subgroup of a group G of finite index. Let τ be a finite-dimensional complex representation of G on which N operates trivially. Then,*

$$H^i(G, \tau) \cong \left[H^i(N, \mathbb{C}) \otimes \tau\right]^G.$$

Proof. By the previous lemma,

$$H^i(G, \tau) \cong H^i(N, \tau)^{G/N} \cong [H^i(N, \mathbb{C}) \otimes \tau]^G.$$ □

This corollary allows us to calculate $H^i(H, \mathbb{C}_{1,\eta})$ as follows.

Corollary 6.4. *For a character $\eta\colon \mathbb{Z}/n \longrightarrow \mathbb{C}^\times$,*

$$H^i(H, \mathbb{C}_{1,\eta}) = \Lambda^i(A^\vee \otimes \mathbb{C})[\eta],$$

where $\Lambda^i(A^\vee \otimes \mathbb{C})[\eta]$, is the η-isotypic component of $\Lambda^i(A^\vee \otimes \mathbb{C})$, for the action of $\mathbb{Z}/n$ as cyclic permutations on A.

Proof. From Corollary 6.3, we have that $H^i(H, \mathbb{C}_{1,\eta}) = H^i(A, \mathbb{C})[\eta^{-1}]$. Since the cohomology of a free abelian group is the exterior algebra on its dual, the corollary follows. □

Theorem 2 in the ramified case now follows from the fact that $A^\vee \otimes \mathbb{C}$ as a module for $\mathbb{Z}/n$ is the sum of all nontrivial characters of $\mathbb{Z}/n$.

7. Preliminaries on Iwahori-Hecke algebras

Now suppose that ω is an unramified character of $k^\times$ of order n, and we are considering the principal series representation $\mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$ of $\mathrm{GL}_n(k)$ restricted to $\mathrm{SL}_n(k)$. In this case the corresponding Hecke algebra governing the situation is the Iwahori-Hecke algebra, which we review below in greater generality than needed for the problem at hand.

Let G be an unramified group, i.e., a quasi-split group over k which splits over an unramified extension of k with $\mathcal{I}$ as an Iwahori subgroup of G, with $\mathcal{I} \subset K$, a hyperspecial maximal compact subgroup of G. Let T be a maximal torus in G which is maximally split, such that $T(\mathcal{O}) \subset \mathcal{I}$. (Recall that since G is unramified, so is T, and hence it makes sense to speak of $T(\mathcal{O})$ which is the maximal compact subgroup of T.) Let $W = N(T)(k)/T(k)$ be the Weyl group associated to the torus T. Let $X_*(T)$ be the cocharacter group of T. Fix a uniformizer ϖ in k, and for a cocharacter μ of T, let ϖ^μ denote the image of ϖ in T under the map $\mu\colon k^\times \to T$. The map $\mu \to \varpi^\mu$ gives an isomorphism of $X_*(T)$ with $T/T(\mathcal{O})$, and hence induces an isomorphism of the group ring $R = \mathbb{Z}[X_*(T)]$ with $\mathcal{H}(T/\!\!/T(\mathcal{O}))$.

We recall (from [9]) that according to the Bernstein presentation of the Iwahori-Hecke algebra $\mathcal{H}(\mathcal{I}) = \mathcal{H}(G/\!\!/\mathcal{I})$, there is the subalgebra $\mathcal{H}(T/\!\!/T(\mathcal{O}))$ generated as a vector space by the elements $\mathcal{I}\varpi^\mu\mathcal{I}$ for μ in the set of coweights; the multiplication is $(\mathcal{I}\varpi^\mu\mathcal{I})(\mathcal{I}\varpi^\nu\mathcal{I}) = \mathcal{I}\varpi^{\mu+\nu}\mathcal{I}$ for μ and ν dominant coweights. The algebra $\mathcal{H}(T/\!\!/T(\mathcal{O}))$ is a Laurent polynomial algebra $\mathbb{Z}[X_*(T)]$. There is also the subalgebra $\mathcal{H}(K/\!\!/\mathcal{I})$ of the Iwahori Hecke algebra consisting of $\mathcal{I}$-bi-invariant functions on G with support in K. The natural map

$$\mathcal{H}(K/\!\!/\mathcal{I}) \otimes \mathcal{H}(T/\!\!/T(\mathcal{O})) \longrightarrow \mathcal{H}(\mathcal{I})$$

is an isomorphism of vector spaces. In particular, $\mathcal{H}(\mathcal{I})$ is a free module over $R = \mathcal{H}(T/\!\!/T(\mathcal{O}))$ of rank equal to the order of W. Furthermore, an irreducible representation of $\mathcal{H}(\mathcal{I})$, when restricted to the commutative subalgebra $\mathcal{H}(T/\!\!/T(\mathcal{O}))$, breaks up as a sum of characters of $\mathcal{H}(T/\!\!/T(\mathcal{O}))$, which are just unramified characters of T which are all conjugate under the action of W. Any character in this Weyl orbit of characters of T is an inducing character for the corresponding unramified principal series representation of G in which this representation of $\mathcal{H}(\mathcal{I})$ is contained; in particular, the unramified principal series representation $\mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$ defines a character ν_ω of $R = \mathcal{H}(T/\!\!/T(\mathcal{O}))$.

It is known that $R^W = \mathcal{H}(T/\!\!/T(\mathcal{O}))^W$ is the center of $\mathcal{H}(\mathcal{I})$, and that if Q denotes the quotient field of R, then the algebra $\mathcal{H}(\mathcal{I}) \otimes_{R^W} Q^W = \mathcal{H}(\mathcal{I}) \otimes_R Q$

is isomorphic to what is called a twisted group ring of W over Q with the natural action of W on R and hence on Q. In fact, we do not need to invert all the nonzero elements of R to get to the twisted group ring, and inverting just one element,

$$\delta = \prod(1 - q^{-1}\varpi^{\alpha^\vee})$$

(where q is the order of the residue field of k, and the product is over all coroots $\alpha^\vee$), is sufficient. Clearly δ is a W-invariant element of R, so belongs to the center $Z = R^W$ of $\mathcal{H}(\mathcal{I})$. Note that δ is not invertible in R as R being a Laurent polynomial algebra, the only invertible elements of R are the monomials.

We now localize $\mathcal{H}(\mathcal{I})$, R, Z at the central multiplicative set given by the powers of δ. Write $\mathcal{H}(\mathcal{I})_\delta$, R_δ, Z_δ for these localizations. The algebra $\mathcal{H}(\mathcal{I})_\delta$ has a simple structure. In fact,

$$\mathcal{H}(\mathcal{I})_\delta = \oplus_{w\in W} R_\delta K_w,$$

where the normalized intertwining operators K_w are as described in [9, §2.2]. Now W acts naturally on R and R_δ and we have

$$K_w r = w(r)K_w \qquad \text{for all } r \in R.$$

We also have

$$K_w K_{w'} = K_{ww'};$$

these equations determine the algebra structure on $\mathcal{H}(\mathcal{I})_\delta$, and prove that $\mathcal{H}(\mathcal{I})_\delta \cong R_\delta[W]$.

Note that from the explicit form of δ given above, $\nu_\omega(\delta) \neq 0$, and hence the character ν_ω of R that we work with extends uniquely to R_δ. We continue to write ν_ω for this extension to R_δ.

8. Proof of Theorem 2 in the unramified case

Let $\omega\colon k^\times \longrightarrow \mathbb{C}^\times$ be an unramified character of order n. Recall that the principal series representation $\pi = \mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$ of $\mathrm{GL}_n(k)$ decomposes as a direct sum $\pi = \sum_\alpha \pi_\alpha$ of n irreducible admissible representations of $\mathrm{SL}_n(k)$ where $\alpha \in k^\times/\ker(\omega)$, all of which have Iwahori-fixed vectors. Extensions between these can therefore be determined through the Iwahori-Hecke algebra $\mathcal{H}(\mathcal{I})$ of G. Since the space of $\mathcal{I}$-invariants in a principal series representation of any split group, in particular $\mathrm{SL}_n(k)$, has dimension equal to the order $|W|$ of the Weyl group W, the representations of the Iwahori-Hecke algebra corresponding to any π_α is of dimension $(n-1)!$ (all being of equal dimension). To justify this, we note that $\dim(\pi_\alpha^{\mathcal{I}})$ is independent of α since:

(1) $\mathrm{GL}_n(k)$ operates transitively on the set of π_α.
(2) If $N(\mathcal{I})$ denotes the normalizer of $\mathcal{I}$ in $\mathrm{GL}_n(k)$, then $N(\mathcal{I})\cdot\mathrm{SL}_n(k) = \mathrm{GL}_n(k)$ since $\mathcal{I}$ is normalized by an element of $\mathrm{GL}_n(k)$ whose determinant is a uniformizer of k. For example, if $\mathcal{I}$ is the "standard"

Iwahori subgroup, then one such element is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \cdots & 0 \\ & & 0 & \ddots & 0 & 0 \\ & & & \ddots & 1 & 0 \\ & & & & 0 & 1 \\ \varpi & & & & & 0 \end{pmatrix}.$$

Using the notation from the previous section, our context consists of the following chain of $\mathbb{C}$-algebras: $R^W \subset R \subset \mathcal{H}(\mathcal{I})$ where R as well as R^W are Laurent polynomial algebras, and R^W is the center of $\mathcal{H}(\mathcal{I})$ which we now abbreviate to $\mathcal{H}$. We have two modules M_1, M_2 over $\mathcal{H}$ which are of dimension $(n-1)!$ over $\mathbb{C}$ arising from two irreducible components of the principal series representation $\mathrm{Ps}(1, \omega, \omega^2, \cdots, \omega^{n-1})$ of $\mathrm{GL}_n(k)$ restricted to $\mathrm{SL}_n(k)$, and we are interested in calculating:

$$\mathrm{Ext}^i_{\mathcal{H}}(M_1, M_2).$$

From results of the previous section, we know that there is an element δ in R^W, such that inclusion $R_\delta \subset \mathcal{H}_\delta$ is the inclusion $R_\delta \subset R_\delta[W]$. We also know from the previous section that the element δ acts invertibly on M_1, and M_2, and therefore M_1 and M_2 can be considered as modules for $\mathcal{H}_\delta = R_\delta[W]$. Since the inclusion $\mathcal{H} \subset \mathcal{H}_\delta$ is flat, generalities from homological algebra imply that:

$$\mathrm{Ext}^i_{\mathcal{H}}(M_1, M_2) \cong \mathrm{Ext}^i_{\mathcal{H}_\delta}(M_1, M_2).$$

Given the inclusion of the twisted group rings $R[W] \subset R_\delta[W]$, let M_1', resp. M_2', be the module M_1, resp. M_2, restricted to $R[W]$. Then we have,

$$\mathrm{Ext}^i_{R[W]}(M_1', M_2') \cong \mathrm{Ext}^i_{R_\delta[W]}(M_1, M_2).$$

The twisted group ring $R[W]$ is the group ring of $A \rtimes S_n$ where

$$A = \left\{ (k_1, \cdots, k_n) \in \mathbb{Z}^n \;\middle|\; \sum k_i = 0 \right\},$$

on which there is the natural action of the symmetric group S_n. The modules M_1' and M_2' can therefore be considered as irreducible representations, say M_1'', M_2'' of $A \rtimes S_n$, and we have

$$\mathrm{Ext}^i_{R[W]}(M_1', M_2') \cong \mathrm{Ext}^i_{A \rtimes S_n}(M_1'', M_2'').$$

Thus we are led to a question about extensions between representations of a group, which in this case is $A \rtimes S_n$. Such questions are well-known to be related to cohomology of groups, using which we will eventually be able to prove that

$$\mathrm{Ext}^i_{A \rtimes S_n}(M_1'', M_2'') \cong \mathrm{Ext}^i_{A \rtimes \mathbb{Z}/n}(\chi_1, \chi_2)$$

where $\mathbb{Z}/n$ is the cyclic group generated by the n-cycle $(1, 2, \cdots, n)$ in S_n, and χ_1, χ_2 are characters of $A \rtimes \mathbb{Z}/n$ which extend the character $\phi\colon (k_1, \ldots, k_n) \mapsto \omega^{k_1}\omega^{2k_2} \cdots \omega^{nk_n}$ of A with the property that,

$$\begin{aligned} M_1'' &= \operatorname{Ind}_{A\rtimes\mathbb{Z}/n}^{A\rtimes S_n} \chi_1, \\ M_2'' &= \operatorname{Ind}_{A\rtimes\mathbb{Z}/n}^{A\rtimes S_n} \chi_2. \end{aligned}$$

The existence of the characters χ_1, χ_2 with the above properties is a simple consequence of Clifford theory since the character of A being considered has stabilizer $\mathbb{Z}/n$ generated by the n-cycle $(1, 2, \cdots, n)$ in S_n.

Thus our calculations made in §6 for the totally ramified case become available, proving Theorem 2. To carry out this outline, we begin with some simple generalities.

Lemma 8.1. *Let G be a group, and V a $\mathbb{C}[G]$-module. Assume that there is an element z of the center of G which operates by a scalar $\lambda_z \neq 1$ on V. Then $H^i(G, V) = 0$ for all $i \geq 0$.*

Proof. The proof of this well known lemma depends on the observation that there is a natural action of G on $H^i(G, V)$ in which $g \in G$ acts on G by conjugation, and which on coefficients V acts by $v \to g^{-1}v$. This action of G on $H^i(G, V)$ is known to be the identity, cf. Proposition 3 on page 116 of [21]. On the other hand, the element z in the center of G operates on $H^i(G, V)$ by $\lambda_z^{-1} \neq 1$, proving the lemma. □

Using this, we have:

Proposition 8.2. *Let A be a finitely-generated free abelian group on which $\mathbb{Z}/n$ operates. Let $H = A \rtimes \mathbb{Z}/n$. Then for an irreducible finite-dimensional complex representation V of H, $H^i(H, V) = 0$ unless A acts trivially on V.*

Proof. Note that by Clifford theory, the representation V is obtained as induction of a character χ of a subgroup H' of H containing A, i.e., $V = \operatorname{Ind}_{H'}^{H} \chi$. By Shapiro's lemma, $H^i(H, V) = H^i(H', \chi)$. The proof is then clear by using the previous lemma (applied to $G = A$, an abelian group!) combined with Lemma 6.2. □

We come now to the main proposition needed for our work. Let

$$A = \Big\{ (k_1, \cdots, k_n) \in \mathbb{Z}^n \;\Big|\; \sum k_i = 0 \Big\},$$

on which there is the natural action of the symmetric group S_n which contains the n-cycle $(1, 2, \cdots, n)$, so the group $\mathbb{Z}/n$ generated by this cycle too operates on A. This allows one to construct groups $H = A \rtimes \mathbb{Z}/n$ and $\tilde{H} = A \rtimes S_n$. Let

$$\phi\colon (k_1, \ldots, k_n) \mapsto \omega^{k_1}\omega^{2k_2} \cdots \omega^{nk_n}.$$

be the character of order n of A as before; as noted earlier, the character ϕ of A is invariant under the cyclic permutation action of $\mathbb{Z}/n$ on A.

Proposition 8.3. *Let χ_1 and χ_2 be any two extensions of the character ϕ of A to characters of $H = A \rtimes \mathbb{Z}/n$. Call M_1, resp. M_2, the representation of $\tilde{H} = A \rtimes S_n$, obtained by inducing the characters χ_1, χ_2 of H. Then*

$$\mathrm{Ext}^i_{\tilde{H}}(M_1, M_2) \cong \mathrm{Ext}^i_H(\chi_1, \chi_2).$$

Proof. We recall the generality that

$$\mathrm{Ext}^i_{\tilde{H}}(M_1, M_2) \cong H^i(\tilde{H}, M_1^\vee \otimes M_2).$$

Since $M_j = \mathrm{Ind}_H^{\tilde{H}} \chi_j$ (for $j = 1, 2$), we have

$$M_1^\vee \otimes M_2 \cong \mathrm{Ind}_H^{\tilde{H}}(\chi_1^{-1} \otimes M_2|_H).$$

By Shapiro's lemma it follows that

$$H^i(\tilde{H}, M_1^\vee \otimes M_2) = H^i(H, \chi_1^{-1} \otimes M_2|_H).$$

Since the stabilizer of the character ϕ of A is the group $H = A \rtimes \mathbb{Z}/n$, the restriction of the representation M_2 to A consists of all *distinct* conjugates of the character ϕ under the symmetric group S_n (with $\mathbb{Z}/n$ as the isotropy of ϕ).

Thus the part of the representation $\chi_1^{-1} \otimes M_2|_H$ of H on which A acts trivially is nothing but the one-dimensional representation $\chi_1^{-1}\chi_2$ of H. By Proposition 8.2 it follows that

$$H^i(H, \chi_1^{-1} \otimes M_2|_H) = H^i(H, \chi_1^{-1}\chi_2).$$

Again noting the generality

$$\mathrm{Ext}^i_H(\chi_1, \chi_2) \cong H^i(H, \chi_1^{-1}\chi_2),$$

the proposition is proved. □

9. A question of compatibility

Theorem 2 has been stated after fixing an arbitrary base point, called π_1, among the irreducible components of the principal series representation $\mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$, which gives rise to a parametrization of all components as $(\pi_1)^{\langle e \rangle} = \pi_e$ for $e \in k^\times / \ker(\omega)$ by inner-conjugation action of $k^\times$ on $\mathrm{SL}_n(k)$. On the other hand, the Hecke algebras, eventually identified to the group algebra of $A \rtimes \mathbb{Z}/n$ in the ramified case, and of $A \rtimes S_n$ in the unramified case, give rise to their own parametrizations. The question arises: how do we relate these two very different looking parametrizations?

Recall that a character of A determines an unramified principal series representation of $\mathrm{SL}_n(k)$. Each such character is contained in an irreducible representation of $A \rtimes \mathbb{Z}/n$. When the character of A has n distinct conjugates under the action of $\mathbb{Z}/n$, one constructs this latter representation via induction to $A \rtimes \mathbb{Z}/n$, and there are no choices to be made: the character of A uniquely determines the irreducible representation of $A \rtimes \mathbb{Z}/n$ to which it belongs. However, in our case the character of A is invariant under the action of $\mathbb{Z}/n$, so it extends in n distinct ways to $A \rtimes \mathbb{Z}/n$. These extended characters of $A \rtimes \mathbb{Z}/n$ are permuted transitively by multiplication by characters of $\mathbb{Z}/n$ since $\mathbb{Z}/n$ is a quotient of $A \rtimes \mathbb{Z}/n$.

The following proposition answers the question of compatibility. We let $G = \mathrm{SL}_n(k)$ below.

Proposition 9.1. *For $e \in k^\times/\ker(\omega)$, the map $\chi \mapsto \chi(e)$ establishes an identification of the character group of $\{1, \omega, \cdots, \omega^{n-1}\} = \mathbb{Z}/n$ with $k^\times/\ker(\omega)$. Fix an irreducible summand π_1 of the principal series representation $\mathrm{Ps}(1, \omega, \cdots, \omega^{n-1})$ of $\mathrm{SL}_n(k)$. For ω a ramified character, the corresponding character of the Hecke algebra $\mathcal{H}(G, \rho)$, corresponds to a character —call it χ_0— of $A \rtimes Z/n$. Then the representation of $\mathcal{H}(G, \rho)$ corresponding to the character $\chi_0 \cdot \chi$ of $A \rtimes \mathbb{Z}/n$ is the same as the one corresponding to $\pi_{e(\chi)}$. In the unramified case, if π_1 corresponds to $\mathrm{Ind}_{A \rtimes \mathbb{Z}/n}^{A \rtimes S_n}(\chi_0)$, then $\pi_{e(\chi)}$ corresponds to $\mathrm{Ind}_{A \rtimes \mathbb{Z}/n}^{A \rtimes S_n}(\chi_0 \cdot \chi)$.*

The proof of this proposition depends on the following simple lemma, whose proof is omitted.

Lemma 9.2. *Let C be a finite cyclic group of order n, and ω a character $C \to \mathbb{C}^\times$. Then ω extends to a character $\tilde{\omega}\colon \mathbb{Z}[C] \to \mathbb{C}^\times$ by sending an element c of C to $\omega(c)$. The restriction of $\tilde{\omega}$ to the augmentation ideal $\mathbb{Z}[C]^0$ is invariant under the translation action of C on $\mathbb{Z}[C]^0$. Thus it extends to a character, say $\tilde{\omega}_0$, of $\mathbb{Z}[C]^0 \rtimes C$. Since $\mathbb{Z}[C]^0 \rtimes C$ is a normal subgroup of $\mathbb{Z}[C] \rtimes C$, there is an action of $[\mathbb{Z}[C] \rtimes C]/[\mathbb{Z}[C]^0 \rtimes C] = \mathbb{Z}$ on $\mathbb{Z}[C]^0 \rtimes C$, and hence on its character group. Under this action, the element $d \in \mathbb{Z}$ takes $\tilde{\omega}_0$ to $\tilde{\omega}_0 \cdot \omega^d$ where ω^d is a character of C thought of as a character of $\mathbb{Z}[C] \rtimes C$.*

Proof of Proposition 9.1. In both the ramified and unramified cases, we will embed our Hecke algebra $\mathcal{H}(G, \rho)$ for $\mathrm{SL}_n(k)$ into a similar Hecke algebra for $\mathrm{GL}_n(k)$.

In the case where ω is totally ramified, the type (K, ρ) for $\mathrm{SL}_n(k)$ has a natural variant for $\mathrm{GL}_n(k)$ with the type $(k^\times \cdot K, \rho')$, where ρ' is the extension of the representation ρ of K to $k^\times \cdot K$ by using the central character of the principal series representation π on $k^\times$.

In the case where ω is unramified, consider the chain of groups $\mathrm{SL}_n(k) \subset k^\times \cdot \mathrm{SL}_n(k) \subset \mathrm{GL}_n(k)$, and the corresponding Iwahori subgroups $\mathcal{I} \subset \mathcal{O}^\times \cdot \mathcal{I} \subset \tilde{\mathcal{I}}$. We can embed the Iwahori algebra $\mathcal{H}(\mathcal{I})$ of $\mathrm{SL}_n(k)$ into the analogous algebra $\mathcal{H}(k^\times \cdot \mathrm{SL}_n(k) /\!\!/ \mathcal{O}^\times \cdot \mathcal{I})$. We will then compare this latter Hecke algebra with the Iwahori-Hecke algebra $\mathcal{H}(\tilde{\mathcal{I}})$ of $\mathrm{GL}_n(k)$.

Recall that instead of considering $\mathcal{H}(\mathcal{I})$ we are considering $\mathcal{H}(\mathcal{I})_\delta$, obtained by inverting an element δ of its center, which can be related to the group algebra of $A \rtimes S_n$. A similar assertion for $\mathrm{GL}_n(k)$ allows one to turn questions on Hecke algebras for $\mathrm{GL}_n(k)$ to one on affine Weyl group for $\mathrm{GL}_n(k)$.

The affine Weyl groups for $k^\times \cdot \mathrm{SL}_n(k)$ and $\mathrm{GL}_n(k)$, parametrizing the double cosets

$$\mathcal{I}\backslash(k^\times \cdot \mathrm{SL}_n(k))/\mathcal{O}^\times \cdot \mathcal{I} \qquad \text{and} \qquad \tilde{\mathcal{I}}\backslash\mathrm{GL}_n(k)/\tilde{\mathcal{I}},$$

respectively, can be identified with

$$(A + \Delta\mathbb{Z}) \rtimes S_n \qquad \text{and} \qquad \mathbb{Z}^n \rtimes S_n,$$

respectively, where $\Delta\mathbb{Z}$ denotes the image of $\mathbb{Z}$ under the diagonal embedding $\Delta\colon \mathbb{Z} \longrightarrow \mathbb{Z}^n$. Consider the short exact sequence

$$0 \longrightarrow (A + \Delta\mathbb{Z}) \rtimes S_n \longrightarrow \mathbb{Z}^n \rtimes S_n \longrightarrow \mathbb{Z}/n \longrightarrow 0, \tag{$*$}$$

with the natural map from $\mathbb{Z}^n \rtimes S_n$ to $\mathbb{Z}$ being the sum of co-ordinates on $\mathbb{Z}^n$. Thus there is a natural action of $\mathbb{Z}^n \rtimes S_n$ on $A \rtimes S_n$ via inner-conjugation, hence on irreducible representations of $A \rtimes S_n$ by inner-conjugation, giving rise to an action of $\mathbb{Z}/n$ on irreducible representations of $A \rtimes S_n$.

The proof of the proposition in the unramified case now follows from Lemma 9.2, applied to the exact sequence

$$0 \longrightarrow (A + \Delta\mathbb{Z}) \rtimes \mathbb{Z}/n \longrightarrow \mathbb{Z}^n \rtimes \mathbb{Z}/n \longrightarrow \mathbb{Z}/n \longrightarrow 0,$$

which is the restriction of the exact sequence $(*)$ to the subgroup $\mathbb{Z}/n$ inside S_n. We leave the details, as well as the case of ramified character, to the reader. We only add that in the ramified case one identifies the Hecke algebra for $\mathrm{GL}_n(k)$ for the type $(k^\times \cdot K, \rho')$ mentioned earlier in the section to the group algebra of $\mathbb{Z}^n \rtimes \mathbb{Z}/n$ such that the previous short exact sequence applies, and together with Lemma 9.2, gives the proof of the proposition. □

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